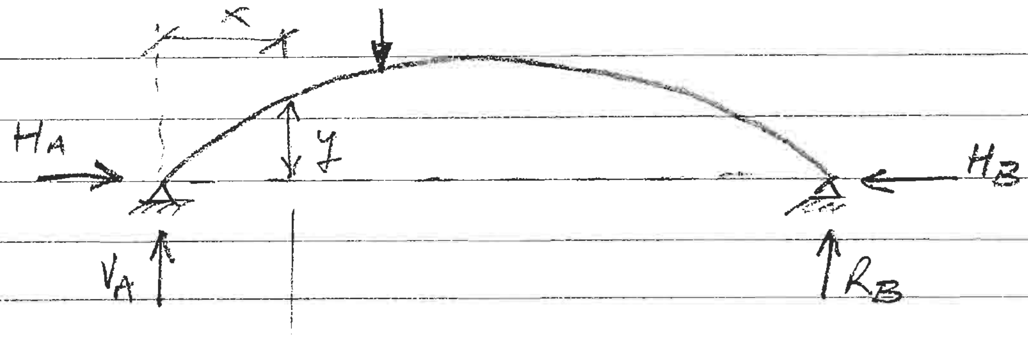
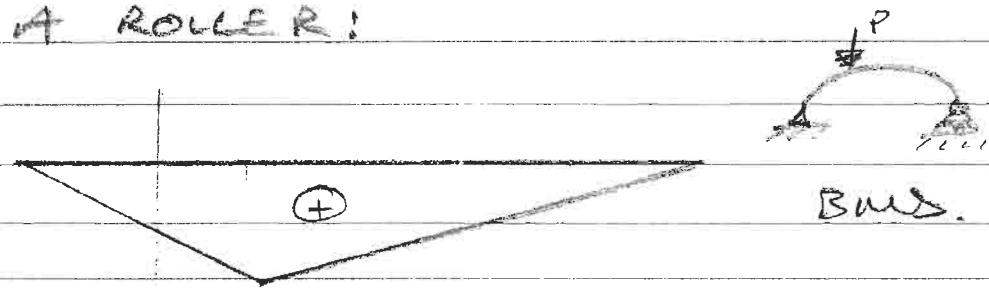
TWO HINGED ARCHES



THE ABOVE ARCH IS INDET. TO 1°.
CUT BACK STRUCTURE BY MAKING SUPPORT AT B A ROLLER:



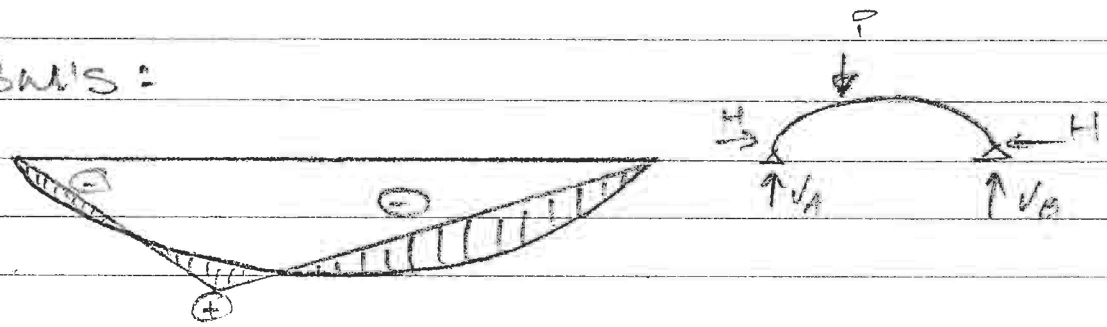
NOW APPLY THE REACTION $H = H_A = H_B$:

SIGN CONVENTION:

⊕ BM - TENSION IN BOTTOM

⊖ BM - TENSION IN TOP.

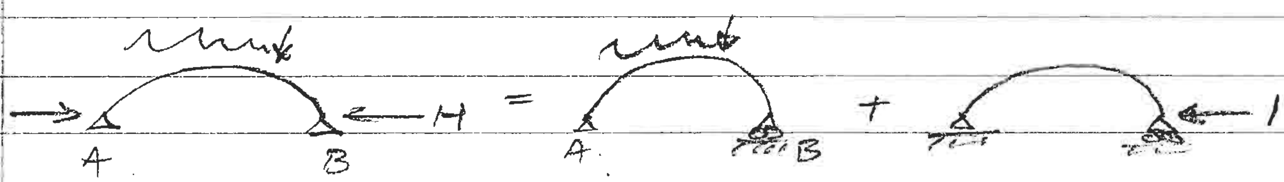
ADD BMS:



NET MOMENT SHOWN HATCHED.

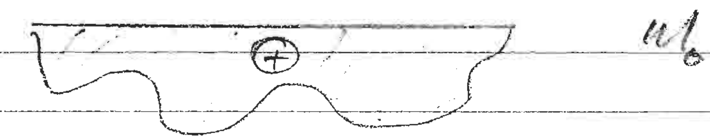
THE EFFECT OF THE HORIZONTAL IS CLEARLY SEEN, THERE IS A LARGE REDUCTION IN THE BM. IN PRACTICE THOUGH, IT IS DIFFICULT TO FIND AN IMMOVABLE HORIZONTAL SUPPORT, THUS ROCK ABUTMENTS ARE VERY ADVANTAGEOUS.

VIRTUAL WORK ANALYSIS

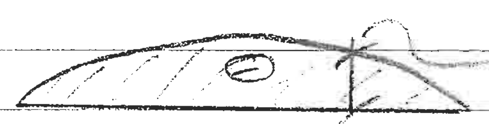


STRUCTURE	=	RELEASED	+	REACTANT
M		M_0		M_1

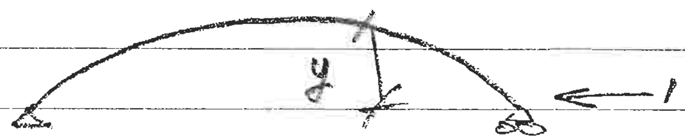
M_0: ORDINARY STAT. DET. SIMPLY SUPPORTED BEAM MOMENTS:



M_1: DUE TO UNIT LOAD AT B:



$M_1 = 1 \times y = y$



THE HORIZONTAL REACTION IS H

$\Rightarrow M = M_0 + H M_1$

EXT. V.W = $1 \times \Delta_B = 1 \times 0 = 0$

INT. V.W = $\int M_1 \left(\frac{M\,ds}{EI} \right)$

EXT V.W = INT. V.W

$\Rightarrow \quad 0 = \int M_1 \left(\frac{M\,ds}{EI} \right)$

BUT $M = M_0 + H M_1$

$\Rightarrow \int \frac{M_1 M_0\,ds}{EI} + H \int \frac{M_1^2\,ds}{EI} = 0$

$\Rightarrow \quad H = \frac{-\int \frac{M_1 M_0\,ds}{EI}}{\int M_1^2\,ds/EI}$

BUT $M_1 = 1x - y = -y$ (TEN ON TOP NEG.)

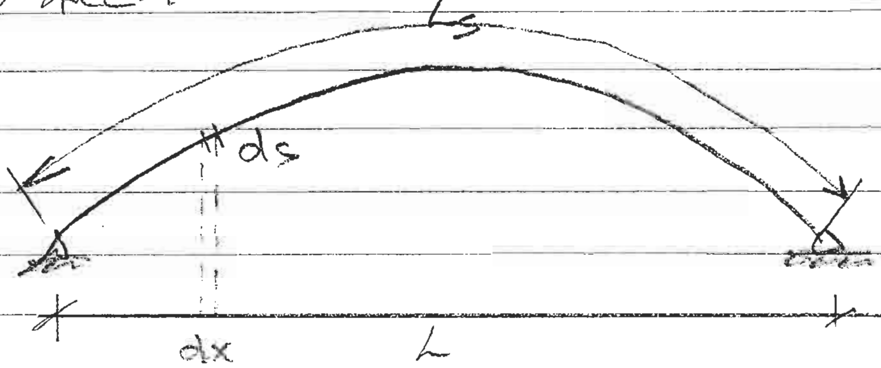
$\Rightarrow \quad H = \frac{+\int M_0 y\,ds/EI}{\int \frac{y^2\,ds}{EI}}$

FOR A PARABOLA: $y = cx(L-x)$

c = CONSTANT, L = LENGTH.

INTEGRAL LIMITS

V.W. EQNS ARE INTEGRATED OVER THE LENGTH OF A MEMBER. IN THE CASE OF AN ARCH:

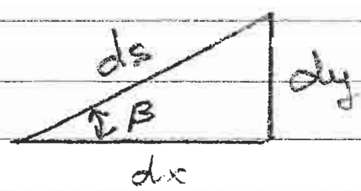


THE INTEGRAL IS OF THE FORM $\int_0^{L_s} ds$

HOWEVER WE PREFER TO USE 'X' OR HORIZONTAL DIMENSIONS:

$$\int_0^{L} dx$$

SO LETS EXAMINE THE RELATIONSHIP BETWEEN dx & ds:



$$\cos\beta = \frac{dx}{ds} \Rightarrow ds = dx \sec\beta$$

THE ANGLE β IS THE ANGLE OF THE TANGENT TO THE CURVE AT ANY POINT x ALONG THE HORIZONTAL.

β IS A MAXIMUM AT THE SUPPORTS AND REDUCES TO ZERO AT THE CROWN OF THE ARCH.

$$\tan \beta = \frac{dy}{dx}$$

WHERE y IS THE EQUATION OF THE ARCH.

CLEARLY TO INCORPORATE THE ANGLE β WOULD REQUIRE AN EVALUATION OF THE EQUATION FOR β AND THEN AN INTEGRATION FROM β TO 0 IS REQ'D:

$$\beta(x) = \tan^{-1} \frac{dy}{dx}$$

$$\Rightarrow \int_{\beta}^{0} \beta(x)\, dx$$

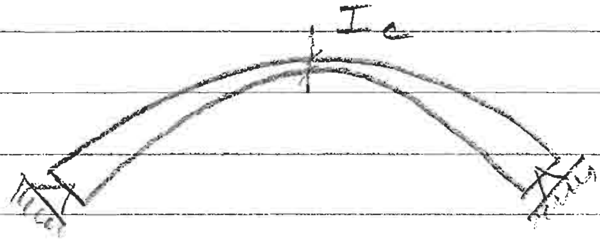
$\Rightarrow$ COMPLEX!!

TO SIMPLIFY, LETS LOOK AT EQN. FOR H WITH SEC β INCLUDED

$$H = \frac{-\int_0^{L_s} \frac{M_1 M_0\,dx}{EI}\sec\beta}{\int_0^{L_s} \frac{M_1^2\,dx}{EI}\sec\beta}$$

WE CANNOT EVALUATE THE INTEGRALS NOW (WITHOUT COMPLEXITY). ALSO WE CANNOT DIVIDE ABOVE AND BELOW BY SEC β AS β IS A FUNCTION OF x.

USUALLY ARCHES ARE OF THE FORM:



$$I(x) = f(x) = f(\beta)$$

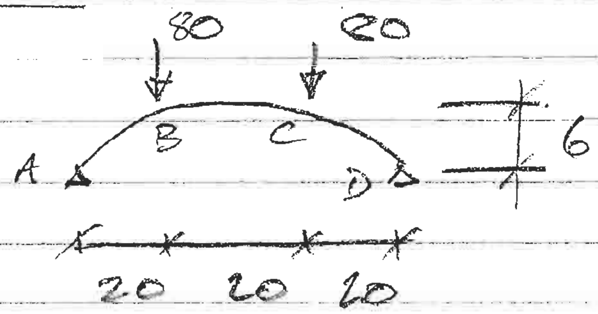
APPROXIMATELY, I WILL VARY ACCORDING TO THE FOLLOWING:

$$I(x) = I_c \sec\beta$$

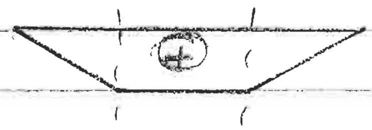
WHERE I_c = I AT CROWN ($\sec\beta = 1$)

NOW WE HAVE A SEC β ABOVE THE LINE AND I_c SEC β BELOW. THE SEC β IS CANCEL AND WE USE I_c IN OUR CALCULATIONS. SIMPLE!!

EXAMPLE 1



M_0



$V_A = V_B = 80$ kN

$\Rightarrow$ FOR $|AB|$ & $|CD|$ $M_0 = 80x$ kNm

FOR $|BC|$ $M_0 = 80 \times 20 = 1600$ kNm

M_1

- DETERMINE EQN OF PARABOLA:

$$y = cx(L - x)$$

AT $x = 30$, $y = 6$ (ABOVE)

$$\Rightarrow 6 = c(30)(60 - 30)$$

$$\Rightarrow c = 1/150$$

$$\Rightarrow y = \frac{x}{150}(L - x)$$

AND THUS $M_1 = \frac{x}{150}(L - x) = y$.

$$H = \frac{\int M_o y \, ds/EI}{\int y^2 ds/EI}$$

- $\int \frac{M_o y ds}{EI}$:

$(AB + CD) = 2\int_0^{20} 80x \left\{\frac{x}{150}(60-x)\right\} dx \quad +$

$(BC) \quad \int_{20}^{40} 1600 \left\{\frac{x}{150}(60-x)\right\} dx$

$= 2\int_0^{20} \frac{8}{15}(60x^2 - x^3)\,dx + \int_{20}^{40} \frac{160}{15}(60x - x^2)\,dx$

$= \frac{16}{15}\left(20x^3 - x^4/4\right)\Big|_0^{20} + \frac{32}{3}\left(30x^2 - x^3/3\right)\Big|_{20}^{40}$

$= 128{,}000 + 184{,}888 = \underline{312{,}888}$

- $\int y^2 ds/EI$:

$= \int [cx(L-x)]^2 dx$

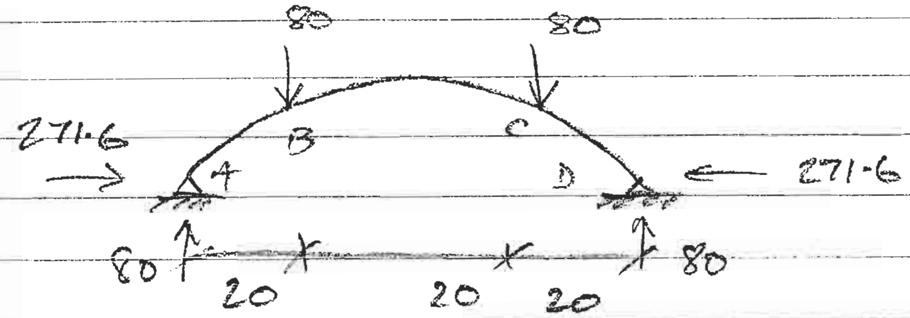
$= \left(\frac{1}{150}\right)^2 \int [60x - x^2]^2 dx$

$= (1/150)^2 \int [60^2x^2 - 120x^3 + x^4]\,dx$

$= (1/150)^2 \left[1200x^3 - 30x^4 + x^5/5\right]_0^{60}$

$= 1152$

$$\Rightarrow H = \frac{312,888}{1152} = \underline{271.6\,kN.}$$

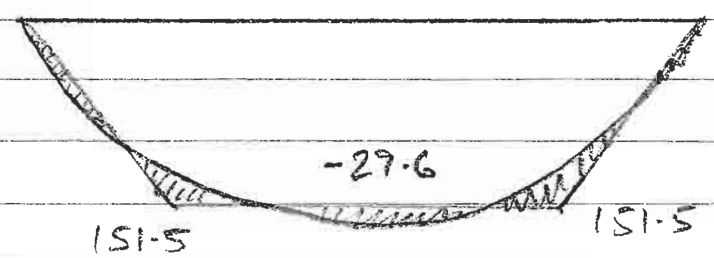


M_B: $y(B) = \frac{1}{150} \times 20(60-20) = 5.33\,m$

$$\Rightarrow M_B = 80 \times 20 - 271.6 \times 5.33$$
$$= 151.5\,kNm$$

$$M_{crown} = 1600 - 271.6 \times 6$$
$$= -29.6\,kNm.$$

PLOT:



(NET MOMENT SHOWN HATCHED.)

PLOT BMD ON HORIZONTAL AXIS:

- AB & DC:

$$M = 80x - 271.6y$$

But $y = 1/150 \times (60 - x) = 0.4x - 0.0067x^2$

$$\Rightarrow \quad M = 80x - 108.64x + 1.81x^2$$
$$= 1.81x^2 - 28.64x$$

$dM/dx = 0$ at max $\Rightarrow 3.62x - 28.64 = 0$

$\Rightarrow M_{max}$ @ $x = 28.64/3.62 = \underline{7.91\,m}$.

- $M_{max} = 1.81(7.91)^2 - 28.64(7.91) = \underline{-113.3\,kNm}$.

- $M = 0$ @ $1.81x^2 - 28.64x = 0$

$\Rightarrow x = 28.64/1.81 = \underline{15.82\,m}$.

- BC:

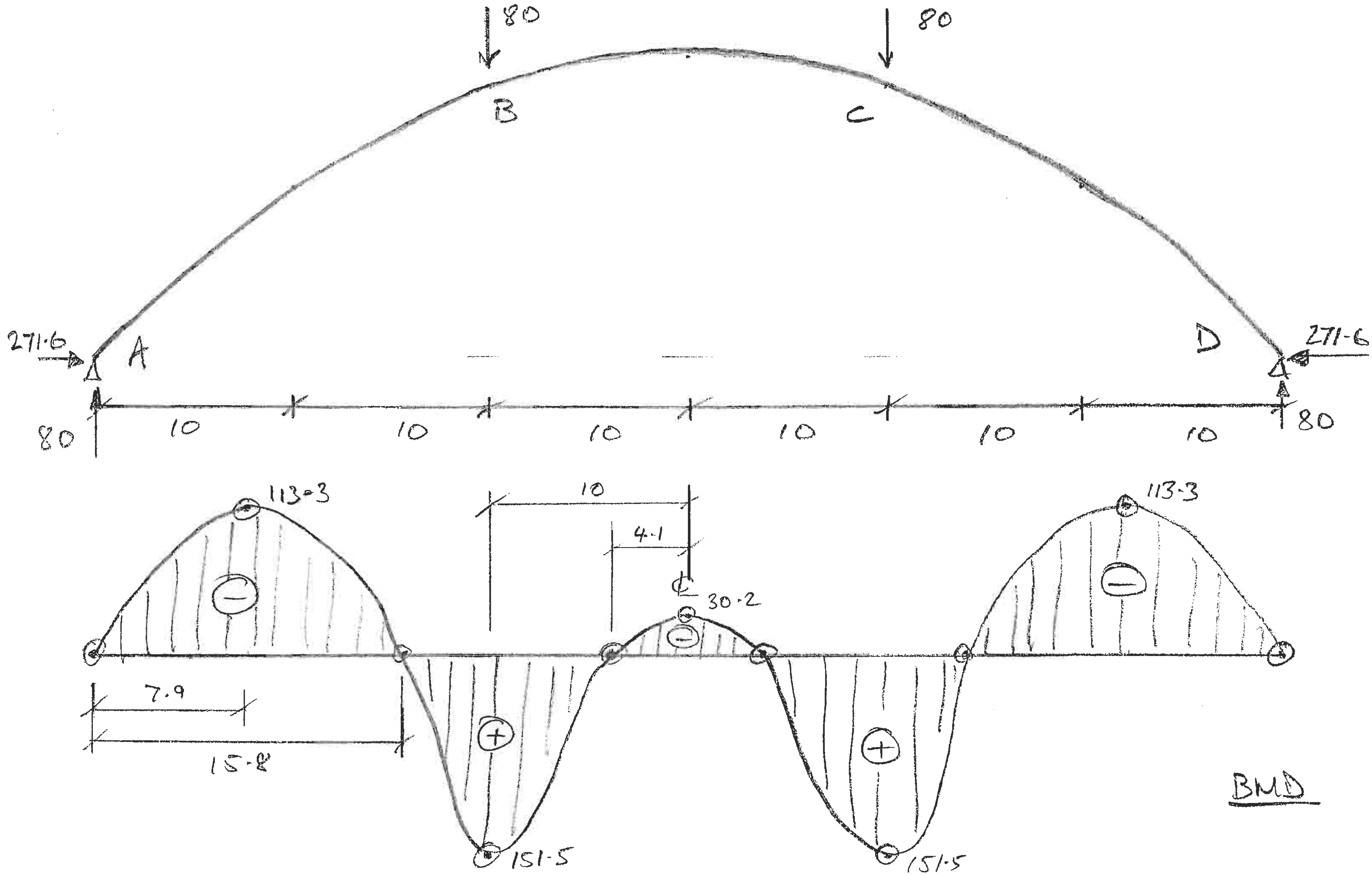
$$M = 1600 - 271.6y$$
$$= 1600 - 108.64x + 1.81x^2$$

$dM/dx = -108.64 + 3.62x = 0 \Rightarrow \underline{x = 30.0\,m}$.

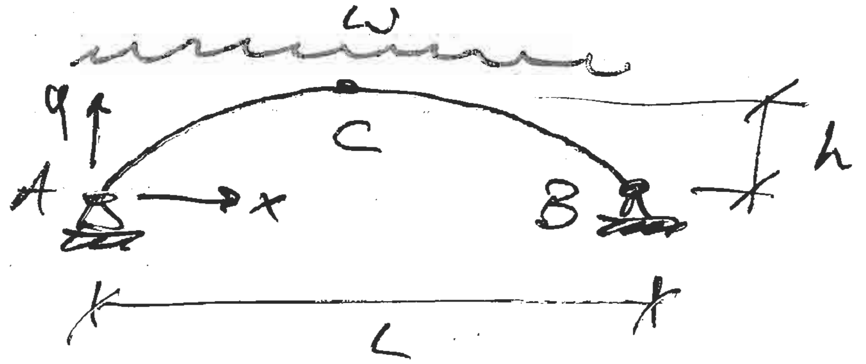
$M_{max} = 1600 - 108.64(30) + 1.81(30)^2 = \underline{-30.2\,kNm}$

$M = 0$ @ $1.81x^2 - 108.64x + 1600 = 0$

$\Rightarrow$ @ $\underline{x = 25.9\,m}$ & $\underline{34.1\,m}$


80
80
B
C
271·6
A
D
271·6
80
10
10
10
10
10
10
80
113·3
10
4·1
113·3
℄
30·2
7·9
15·8
151·5
151·5
BMD

PARABOLIC ARCH WITH APPLIED UDL



Equation of a parabola:

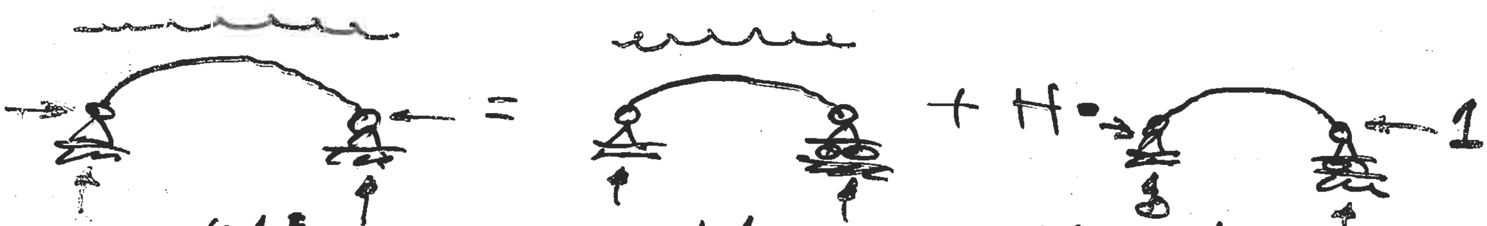
$$y = cx(L - x)$$

But at $x = L/2$, $y = h$

$$\therefore \quad c = 4h/L^2$$

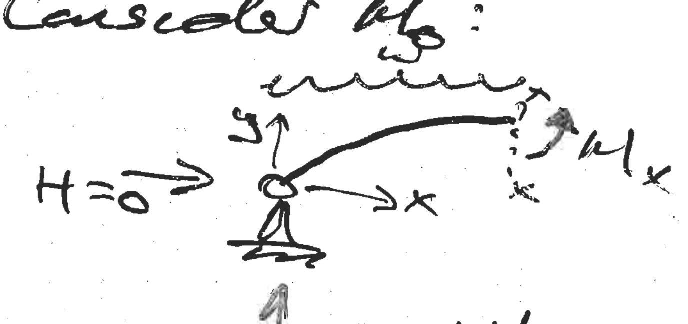
This is the general coefficient for any parabolic arch.

Virtual Work Analysis:



$$M = M_0 + H \cdot m_1$$

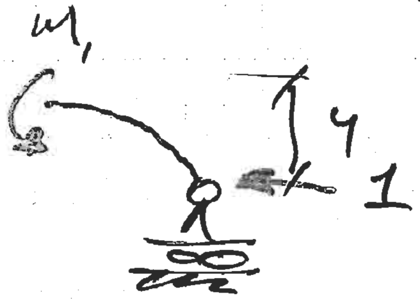
Consider M_0:



$$M_{0,x} = \frac{wL}{2}x - \frac{wx^2}{2}$$

$$V_A = \frac{wL}{2}$$

Consider M_1:



$M_{1,x} = -y(x)$

(−) − tension on top.

$$\Rightarrow \boxed{M_{1,x} = -\frac{4h}{L^2}x(L-x)}$$

The general V.W. Equ. for Arches:

$$\boxed{H = -\frac{\int_0^L \frac{M_1 M_0\,dx}{EI_c}}{\int_0^L \frac{M_1^2\,dx}{EI_c}}}$$

Consider the first term: $\int_0^L \frac{M_1 M_0\,dx}{EI_c}$:

$$\int_0^L \frac{M_1 M_0\,dx}{EI_c} = \frac{1}{EI_c}\int_0^L \left(\frac{wL}{2}x - \frac{wx^2}{2}\right)\left(-\frac{4h}{L^2}x(L-x)\right)dx$$

This simplifies to:

$$\int_0^L \frac{M_1 M_0\,dx}{EI_c} = \frac{1}{EI_c}\cdot\frac{2wh}{L}\int_0^L \left(-Lx^2 + 2x^3 - x^4/L\right)dx$$

From which we get:

$$\boxed{\int_0^L \frac{M_1 M_0\,dx}{EI_c} = \frac{-whL^3}{15EI_c}}$$

The second term:

$$\int_0^L \frac{M_1^2\,dx}{EI_c} = \frac{1}{EI_c}\int_0^L \left[\frac{-4h}{L^2}x(L-x)\right]^2 dx$$

$$= \frac{16h^2}{EI_c L^4}\int_0^L \left(L^2x^2 - 2Lx^3 + x^4\right) dx$$

From which:

$$\boxed{\int_0^L \frac{M_1^2\,dx}{EI_c} = \frac{8h^2L}{15EI_c}}$$ *NB

This last expression is general for any V.W. analysis of a parabolic arch.

Thus:

$$H = \frac{-\left[\frac{-whL^3}{15EI_c}\right]}{\frac{8h^2L}{15EI_c}}$$

$$\therefore \boxed{H = \frac{wL^2}{8h}}$$

Thus the horizontal force in a parabolic arch subjected to a full UDL is equal to the bending moment in the equivalent simply supported beam, divided by the height of the arch.

Also, the bending moments are:

$$M = M_0 + H M_1$$

$$= \frac{wLx}{2} - \frac{wx^2}{2} + \frac{wL^2}{8h}\left[\frac{-4h}{L^2}x(L-x)\right]$$

$$= \frac{wx}{2}\left[L - x - (L-x)\right]$$

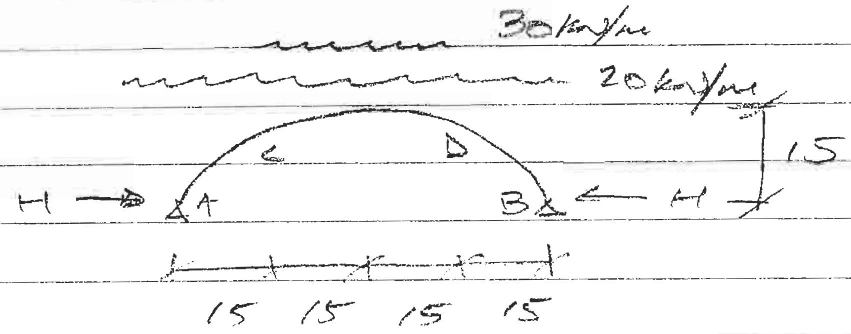
$$= 0$$

There is no bending moment in a parabolic arch subjected to a UDL.

This is a special case of the theory that the most efficient form of a structure (in terms of bending) is that which adopts the same shape of the BMD for an equivalent span or spans of horizontal beam elements.

This is also why post-stressing strands are draped according to the BMD profile.

EXAMPLE 3



FIND M IF SUPPORT B IS JACKED TO GIVE ZERO MOMENT AT CROWN)

$V_A = V_B = (20\times60 + 30\times30)/2 = 1050\,kN$

$$y = cx(L-x)$$

AT $x = 30$, $y = 15$

$\Rightarrow 15 = c\times30(60-30)$

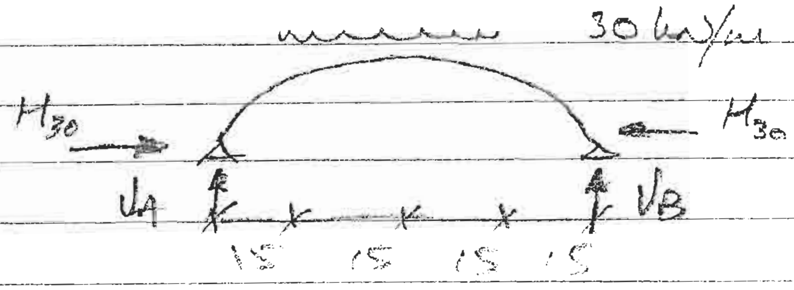
$\Rightarrow c = 1/60$

$$\Rightarrow y = \frac{x}{60}(60-x)$$

WE KNOW THAT THE 20 kN/m UDL CAUSES NO MOMENT IN THE ARCH AND CAUSES $H = wL^2/8h$ AS A HORIZONTAL THRUST:

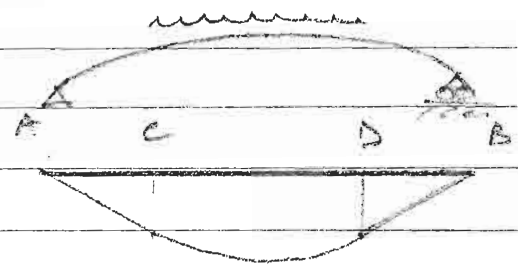
$$H_{20} = \frac{20\times60^2}{8\times15} = 600\,kN.$$

ANY MOMENT IN THE ARCH IS DUE TO THE 30 kN/m LOAD, SO ANALYSE IT USING USUAL METHOD.

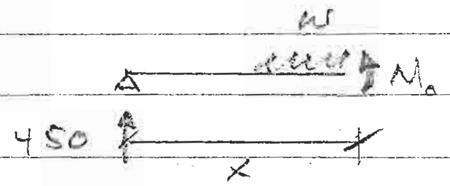


$V_A = V_B = 30 \times 30/2 = 450\,kN.$

- M_o



- |AC| & |DB| $\quad M_o = 450x$
- |CD|



$$M_o = 450x - 30(x-15)^2/2$$
$$= 900x - 15x^2 - 3375$$

- $\int \frac{M_o y\, dx}{EI_c}$

$$= \frac{2}{EI}\int_0^{15} \underbrace{450x\left(\frac{x}{60}\right)(60-x)}_{①}\,dx$$

$$+ \frac{1}{EI}\int_{15}^{45} \underbrace{(-15x^2+900x-3375)\left(\frac{x}{60}\right)(60-x)}_{②}\,dx.$$

SIMPLIFY ①:

$$450x(x/60)(60-x)$$

$$= \frac{450x^2}{60} \cdot 60 - \frac{450x^2}{60} \cdot x$$

$$= 450x^2 - 7.5x^3$$

$$= 7.5[60x^2 - x^3]$$

SIMPLIFY ②:

$$(-15x^2 + 900x - 3375)(x/60)(60-x)$$

$$= \frac{15}{60}(60x - x^2 - 225)(x)(60-x)$$

$$= \frac{1}{4}(3600x^2 - 60x^3 - 13500x - 60x^3 + x^4 + 225x^2)$$

$$= 1/4(3825x^2 - 120x^3 - 13500x + x^4)$$

INTEGRATE:

$$\Rightarrow \frac{2 \times 7.5}{EI}\left[\frac{60x^3}{3} - \frac{x^4}{4}\right]_0^{15}$$

$$+ \frac{1}{4EI}\left[1275x^3 - 30x^4 - 6750x^2 + x^5/5\right]_{15}^{45}$$

$$= \left[2(0.411 \times 10^6) + \frac{1}{4}(16.4 \times 10^6 - 1.41 \times 10^6)\right]\frac{1}{EI}$$

$$= 4.567 \times 10^6/EI$$

$$\int \frac{y^2 dx}{EI} = \frac{1}{EI}\int_0^{60} \left[\left(\frac{x}{60}\right)(60-x)\right]^2 dx.$$

$$= \frac{1}{EI}\int_0^{60} \left(x - x^2/60\right)^2 dx.$$

$$= \frac{1}{EI}\int_0^{60} x^2 - x^3/30 + x^4/3600 \; dx.$$

$$= \frac{1}{EI}\left[\frac{x^3}{3} - \frac{x^4}{120} + \frac{x^5}{18000}\right]_0^{60}$$

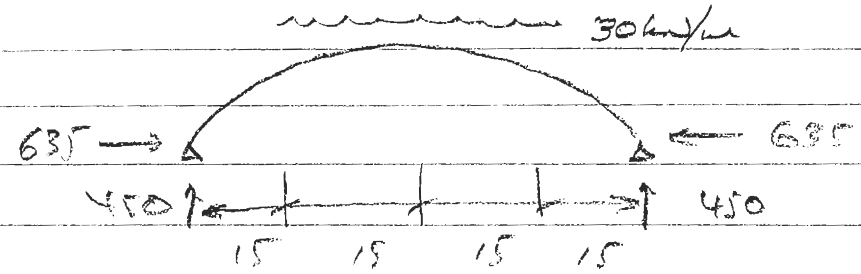
$$= \frac{1}{EI}\left[7200\right]$$

$$H = -\int \frac{M_0 y dx}{EI} \Big/ \int \frac{y^2 dx}{EI}$$

$$= 4.569 \times 10^6 / 7200$$

$$= 635 \text{ kN}.$$

THE RESULT IS: $H = 635\,kN$



THE QUESTION ASKS FOR THE H VALUE THAT GIVES ZERO MOMENT @ CROWN:

$$\text{ZERO MT @ } H_T \Rightarrow H_T = H_{20} + H_{30} + H_J$$

WHERE H_J IS THE JACKING LOAD.

THE ONLY MOMENT IN THE CROWN IS DUE TO THE 30kN/m LOAD:

$$M_{CROWN,30} = 450 \times 30 - 635 \times 15 - 30 \times 15^2/2$$
$$= 600\,kNm.$$

THUS THE JACKING FORCE NEEDS TO PRODUCE A MOMENT OF $-600\,kNm$ FOR ZERO MOMENT @ CROWN:

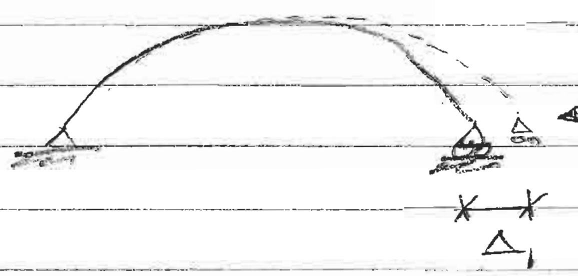
$$\Rightarrow H_J = 600/15 = 40\,kN.$$

TOTAL FORCE/REACTIONS IN SUPPORTS:

$$H_T = H_{20} + H_{30} + H_J$$

$$= 600 + 635 + 40$$
$$= 1275 \text{ kN.} \qquad \text{QED.}$$

THE JACKING LOAD MUST MOVE THE SUPPORT TO PRODUCE A MOMENT IN THE ARCH:



EXT V.W = INT V.W

$$1 \times \Delta_1 = \int \frac{M_1 M dx}{EI} = \int \frac{M_1^2 dx}{EI}$$

$M = M_1$ AS ONLY UNIT LOAD APPLIED.

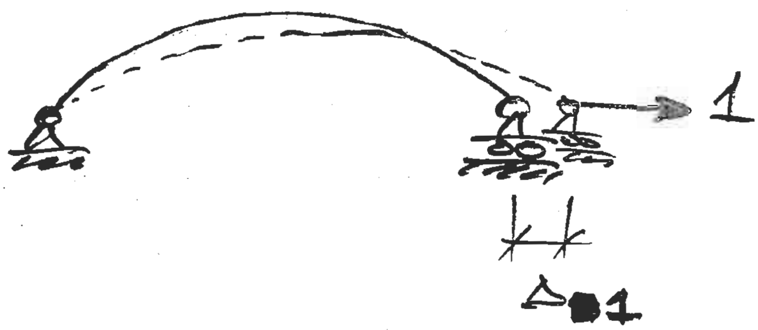
$$\Rightarrow \Delta_1 = \int \frac{M_1^2 dx}{EI} = \int \frac{y^2 dx}{EI} = \frac{7200}{EI}$$

(AS CALCULATED IN THE H_{30} CALCS)

$$\Rightarrow \Delta_{40} = 40 \times \frac{7200}{EI} = \frac{288 \times 10^3}{EI}$$

Displacement of Supports in Arches.

The basic case is:



$$\text{Ext. V.W.} = \text{Int. V.W.}$$

$$\Rightarrow \quad 1 \times \Delta_1 = \int \frac{M_1 M \, dx}{EI_c}$$

But $M = M_0 + H M_1$, $M_0 = 0$, $M_1 = +y(x)$

$$\therefore \quad \Delta_1 = \int \frac{M_1^2 \, dx}{EI_c} = \frac{8h^2 L}{15 EI_c}$$

for a parabolic arch.

i.e. $\frac{8h^2 L}{15 EI_c}$ is the deflection 1 kN would cause

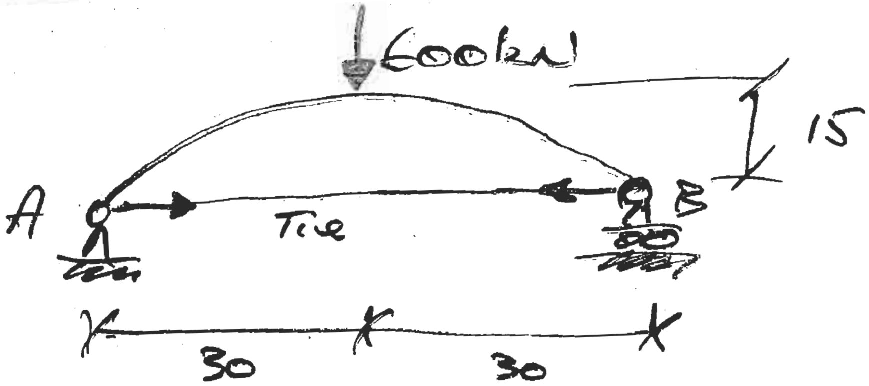
In Ex. 3, the jack needed to jack 40 kN and so would need to travel

$$\Delta_{40} = 40 \times \Delta_1 = 40 \left(\frac{8h^2 L}{15 EI_c} \right)$$

Take $EI_c = 2 \times 10^7 \text{ kNm}^2$

$$\therefore \quad \Delta_{40} = 0.0144 \text{ m} = 14.4 \text{ mm}.$$

EXAMPLE 4 (Q4, S'99)



Calculate:

- Force in tie AB
- BM in arch.
- Δ_{BH}

$I = I_c \sec\theta$

$EI_c = 2\times10^7 \text{ kNm}^2$

$EA = 6\times10^5 \text{ kN}.$

Some observations:

- As B is not pinned, but connected to a cable of finite stiffness, it will deflect.
- B will deflect less than it would if the tie was not present.
- The actual displacement of the arch at B and the elongation of the tie must be one and the same i.e. compatibility

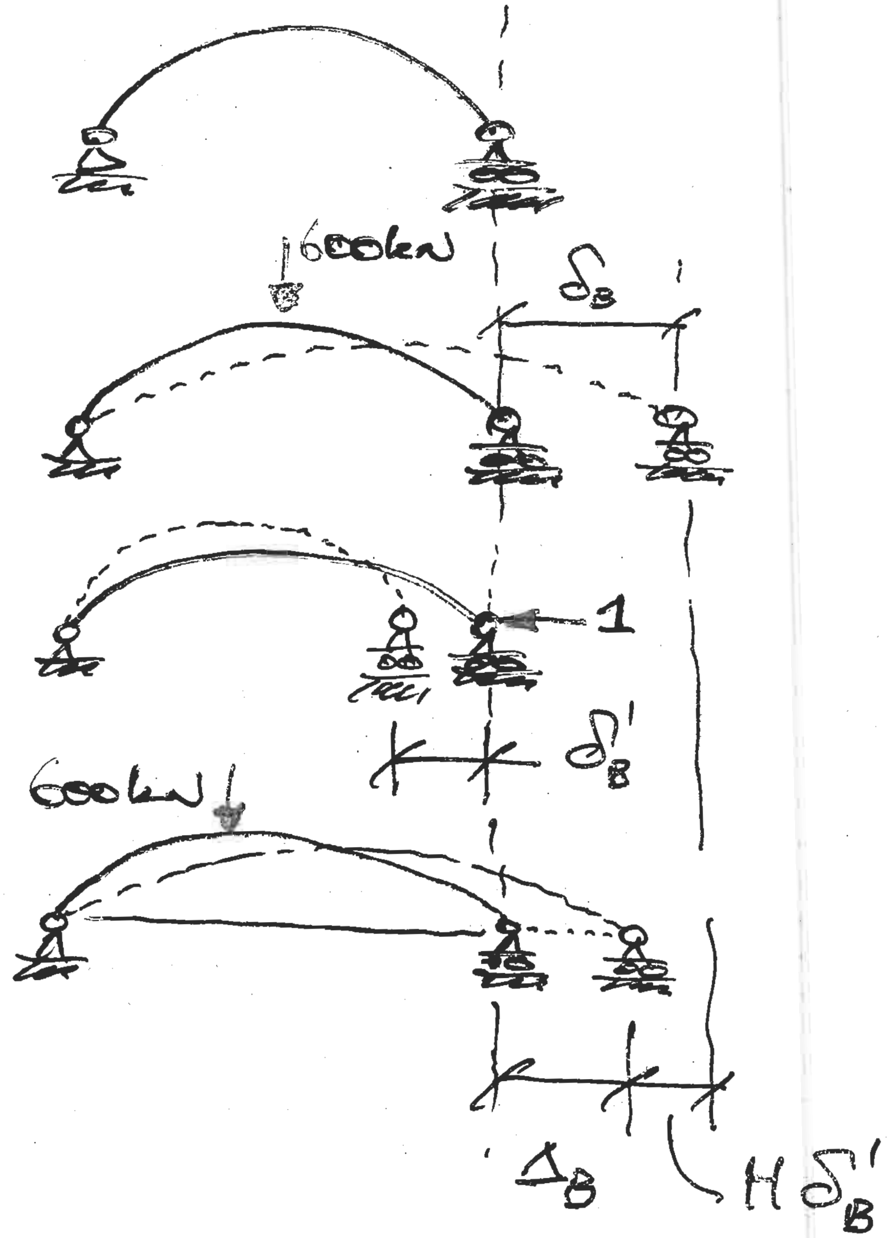
These observations may be drawn as follows and leads to the solution:

Unloaded Arch (No tie) :

~~Loaded Arch~~ LOADED ARCH (No tie) :

Unit load (No tie) :

Actual Arch :



δ_B – disp if no tie present

δ_B' – disp of arch if 1kN in tie

$H\delta_B'$ – disp of arch with H kN in tie

Δ_B – actual disp. of arch and tie

Thus: $\Delta_B = \delta_B - H\delta_B'$ — arch

and $\Delta_B = \frac{HL}{EA}$ ——→ tie

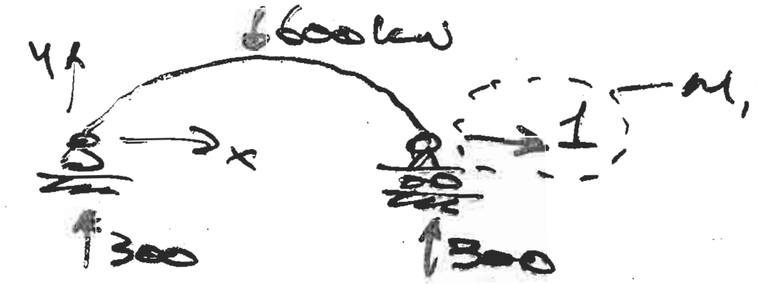
i.e. compatibility of displacement at B.

Thus: $\frac{HL}{EA} = \delta_B - H\delta_B'$

$$\therefore \boxed{H = \frac{\delta_B}{(L/EA + \delta_B')}}$$

We can calculate all terms on the RHS:

$\underline{\delta_B}$



Ext. V.W. = Int. V.W.

$$\therefore 1 \times \delta_B = \int \frac{M_1 M \, dx}{EI_c}$$

But $M = M_0 + HM_1$ but $H = 0$ $\therefore M = M_0$

$$\therefore \delta_B = \int \frac{M_0 M_1 \, dx}{EI_c}$$

$M_1 = y = cx(L-x)$; $c = \frac{4h}{L^2} = 1/60$

$M_0 = 300x$ for $|AC|$

$$\Rightarrow \delta_B = \frac{2}{EI} \int_0^{30} 300x \left[\tfrac{1}{60} x (60-x)\right] dx$$

$$\Rightarrow \boxed{EI_c \delta_B = 3.375 \times 10^6}$$

δ_B'

$$\delta_B' = \int \frac{M_1 M dx}{EI_c}$$

But, $M = M_0 + HM_1$; $M_0 = 0$ & $H = 1$

$$\Rightarrow \delta_B' = \int \frac{M_1^2 dx}{EI_c} = \int \frac{y^2 dx}{EI_c} = \frac{8h^2L}{15EI_c}$$

from results of previous examples

$$\Rightarrow \boxed{EI_c \delta_B' = 7200}$$

L/EA

$$(L/EA)_{tie} = \frac{60}{6\times10^5} = 1\times10^{-4}$$

Thus:

$$H = \frac{3.375\times10^6/EI_c}{\left[1\times10^{-4} + 7200/EI_c\right]}$$

Divide terms by EI_c & solve:

$$H = 364.13\text{ kN}$$

Δ_{BH} :

$$\Delta_B = \frac{HL}{EA} \quad \text{— from previous}$$

$$= 364.13 \times 1\times10^{-4}$$

$$= 36.4\text{ mm}$$

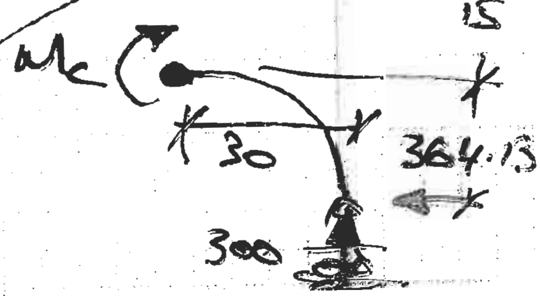
Effect of tie

BMD : Take corner:

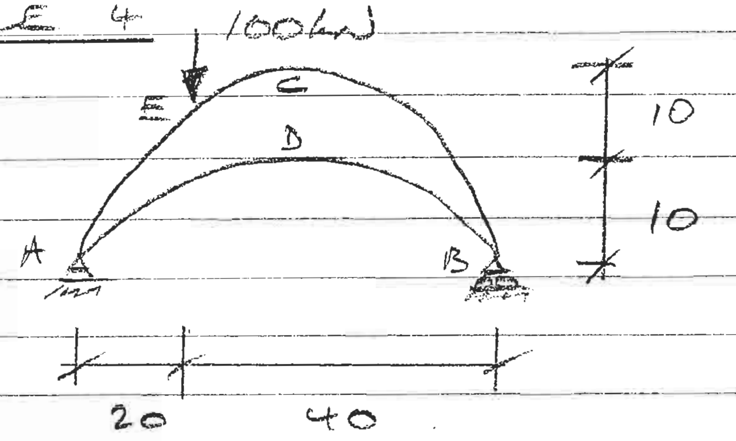
No tie

$$M_c = 300\times30 - 364\times15$$

$$= 3538\text{ kNm}$$



EXAMPLE 4



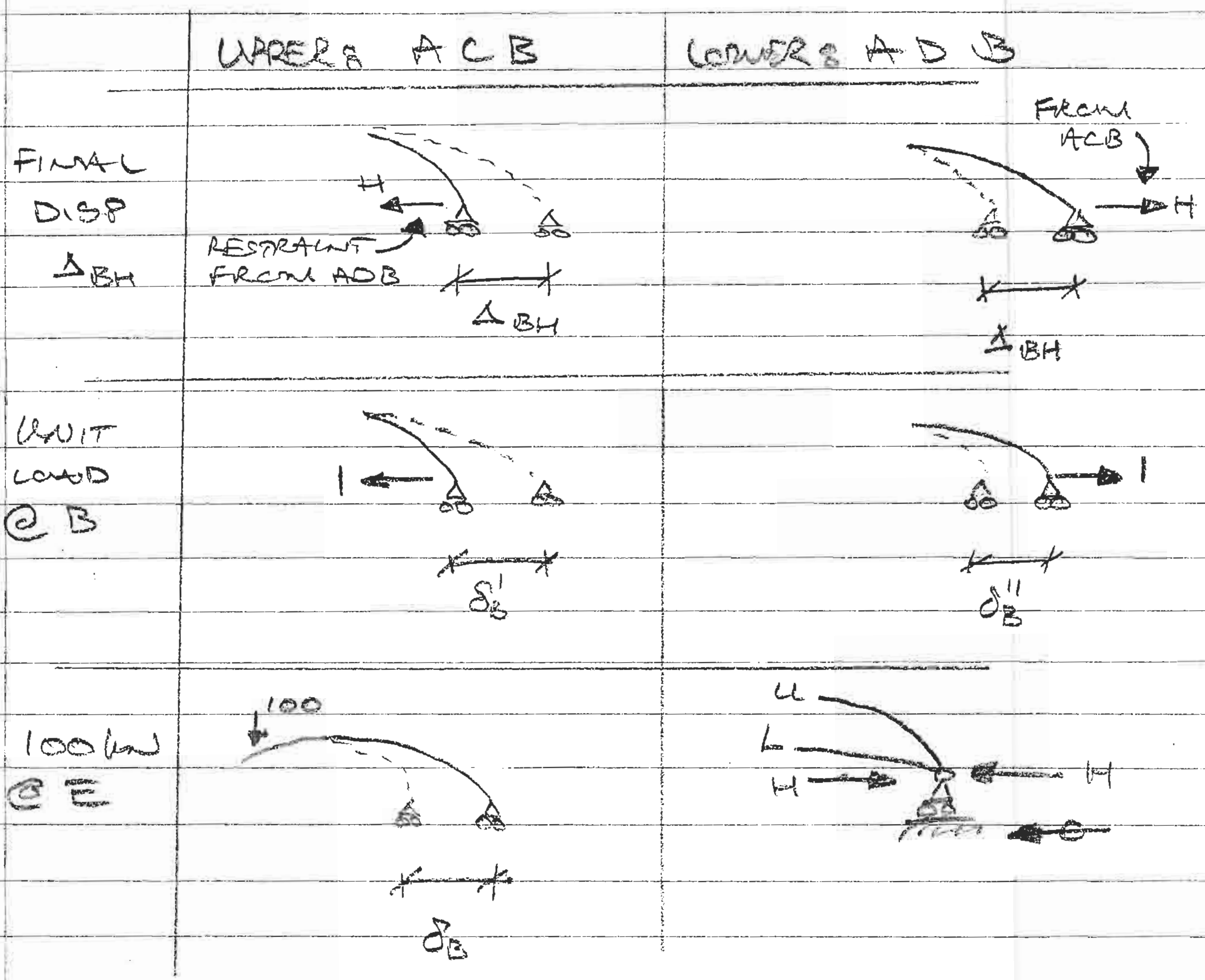
DETERMINE THE MOMENTS AT THE CROWN OF EACH ARCH.

STRUCTURAL BEHAVIOUR

- INITIAL OBSERVATIONS:
  1. BOTH ARCHES CONNECTED @ B
     => COMPATIBILITY OF DISPLACEMENT.
  2. ROLLER SUPPORT @ B
     => NO EXTERNAL H REACTION

SINCE THERE IS NO EXTERNAL H REACTION @ B THE M THAT WOULD HAVE RESULTED @ B DUE TO ARCH ACB IS NOW TAKEN BY ARCH ADB.
=> LOAD/MOMENT IN ARCH ADB

LETS LOOK AT B:



UPPER ARCH

Δ_{BH} = DISP DUE TO 100kN @ E (→)
— DISP DUE TO H @ B (←)

H IS THE HORIZONTAL RESTRAINT PROVIDED BY THE LOWER ARCH

$$\Rightarrow \boxed{\Delta_{BH} = \delta_B - H\,\delta'_B}$$

δ'_B = DSP @ B DUE TO UNIT LOAD @ B FOR THE UPPER ARCH.

LOWER ARCH

Δ_{BH} = DISP DUE TO H AT B (→)

WHERE H IS THE OUTWARD PUSH OF THE UPPER ARCH

$$\Rightarrow \quad \Delta_{BH} = H\,\delta_B''$$

δ_B'' = DISP AT B DUE TO UNIT LOAD @ B FOR THE LOWER ARCH

OVERALL

AS EXTERNAL HORIZONTAL REACTION = 0 THE INWARD PULL BY THE LOWER ARCH MUST = OUTWARD PUSH BY THE UPPER ARCH = H.

ALSO $\quad \Delta_{BH}(ACB) = \Delta_{BH}(ADB)$

$$\Rightarrow \quad \boxed{\delta_B - H\,\delta_B' = H\,\delta_B''}$$

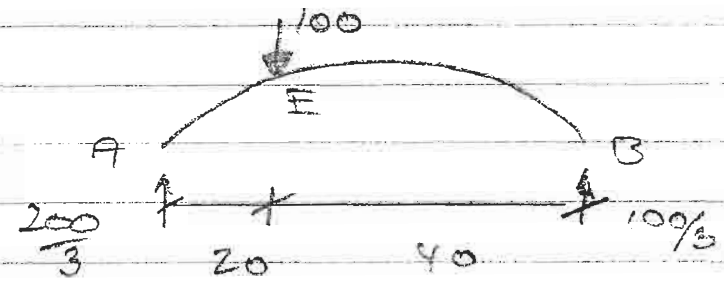
$$\Rightarrow \quad \delta_B = H(\delta_B' + \delta_B'')$$

$$\Rightarrow \quad \boxed{H = \delta_B / (\delta_B' + \delta_B'')}$$

WE CAN CALCULATE δ_B, δ_B', δ_B'' USING VIRTUAL WORK:

VIRTUAL WORK ANALYSIS

• δ_B



$|AE| \rightarrow M = \frac{200}{3}x$

$|BE| \rightarrow M = 100/3 \; x$

$$\text{EXT V.W.} = \text{INT V.W.}$$

$$1 \times \delta_B = \int \frac{M_1 M \, dx}{EI}$$

$$\Rightarrow \delta_B = \int \frac{M y \, dx}{EI} \qquad (y = M_1)$$

$$y = cx(L-x)$$

FOR UPPER ARCH:

AT $x = 30$, $y = 20 = c(30)(60-30) \Rightarrow c = 1/45$

$$\Rightarrow y = \frac{x}{45}(60-x)$$

$$\delta_B = \frac{1}{EI}\left[\int_0^{20}\left(\frac{200x}{3}\right)\left(\frac{x}{45}(60-x)\right)dx + \int_0^{40}\left(\frac{100x}{3}\right)\left(\frac{x}{45}(60-x)\right)dx\right]$$

$$= \frac{1}{EI}\left[\int_0^{20}(88.88x^2 - 1.48x^3)\,dx + \int_0^{40}(44.44x^2 - 0.74x^3)\,dx\right]$$

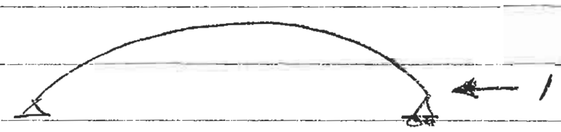
$$= \frac{1}{EI}\left\{\left[29.626x^3 - 0.37x^4\right]_0^{20} + \left[14.81x^3 - 0.18x^4\right]_0^{40}\right\}$$

$$= \frac{1}{EI}\left[177.8\times10^3 + 487\times10^3\right]$$

$$= 651.8\times10^3/EI$$

$$\Rightarrow \quad \underline{\delta_B EI = 651.8\times10^3}$$

- $\underline{\delta_B'}$



$$\delta_B' = \int \frac{M_1 M\,dx}{EI}$$

$$= \int \frac{y^2\,dx}{EI}$$

$$= \frac{8h^2 L}{15EI} \quad \text{(SEE UDL EXAMPLE)}$$

$$= \frac{8\times20^2\times60}{15EI}$$

$$\Rightarrow \quad \underline{\delta_B' EI = 12.8\times10^3}$$

• δ_B''

$$\delta_B'' = \int \frac{M_1 M_1 \, dx}{EI}$$

$$= \int \frac{y^2 \, dx}{EI}$$

$$= \frac{8h^2 L}{15EI}$$

$$= \frac{8 \times 10^2 \times 60}{15EI}$$

$$\Rightarrow \quad \delta_B'' EI = 3.2 \times 10^3$$

NOTE: NOTICE HOW MUCH EFFORT IS SAVED BY KNOWING $\int y^2 dx / EI = 8h^2 L / 15EI$!!

SOLUTION FOR H:

$$H = \frac{\delta_B}{(\delta_B' + \delta_B'')}$$

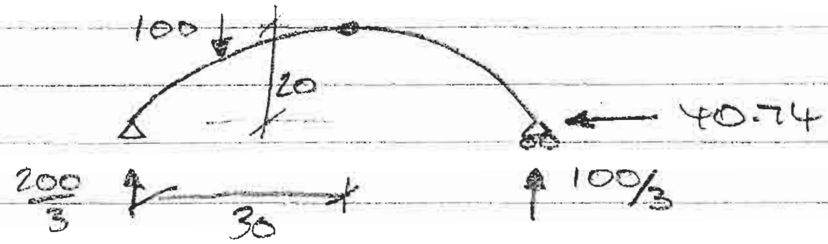
$$= \frac{651.8}{(12.8 + 3.2)}$$

$$= 40.74 \text{ kN}$$

NOTE: NOTICE HOW HANDY HAVING ALL RESULTS TO THE SAME POWER IS ($\times 10^3$)!

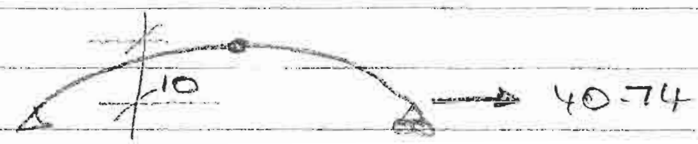
ANSWERS

- **M_{crown} ACB:**



$$M_{crown} = \frac{100}{3} \times 30 - 40.74 \times 20 = \underline{+185 \text{ kNm}}.$$

- **M_{crown} ADB:**



$$M_{crown} = 40.74 \times 10 = \underline{+407.4 \text{ kNm}}$$

⊕ → TENSION IN BTM.

- **Δ_{BH}:**

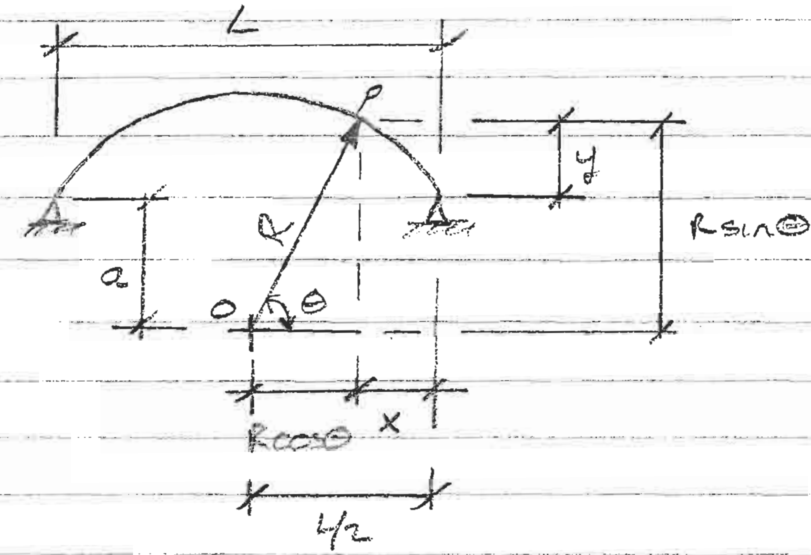
$$\Delta_{BH} = H\,\delta_B''$$

$$= 40.74 \times \frac{3.2 \times 10^3}{EI}$$

$$\Rightarrow \underline{\Delta_{BH} EI = 130.37 \times 10^3}$$

CIRCULAR ARCHES.

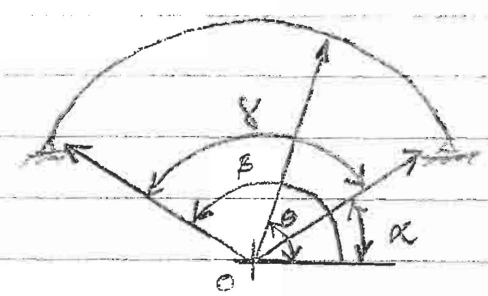
THE ONLY DIFFERENCE IN THE ANALYSIS OF CIRCULAR & PARABOLIC ARCHES IS THE EQUATION OF THE ARCH. WE HAVE ALREADY COVERED THE PARABOLIC CASE. THE CIRCULAR IS:



ANY POINT P MAY BE LOCATED WITH

$$x = L/2 - R\cos\theta \qquad y = R\sin\theta - a$$

INTEGRATION LIMITS ARE:

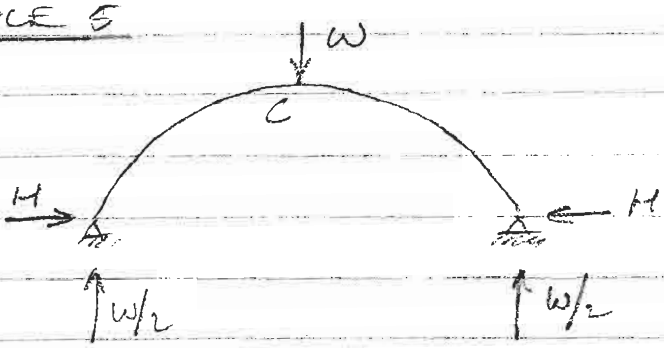


$$\gamma = \beta - \alpha$$

NOTE: ALL ANGLES TO BE IN RADIANS

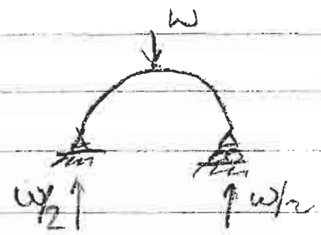
$$\Rightarrow \int_{\alpha}^{\beta} d\theta$$

EXAMPLE 5



Semi Circular Arch
$\Rightarrow a = 0$

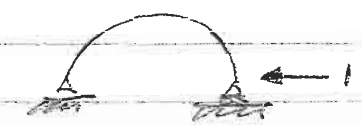
• M_0



$$M_0 = \frac{W}{2}x = \frac{W}{2}\left[\frac{L}{2} - R\cos\theta\right]$$

$$= \frac{W}{2}\cdot R\,(1-\cos\theta)$$

• M_1



$$M_1 = y = R\sin\theta$$

• $$\int \frac{M_0\,y\,ds}{EI} = \frac{2}{EI}\int_0^{\pi/2} \frac{WR}{2}\left[1-\cos\theta\right]\left[R\sin\theta\right](R\cdot d\theta)$$

$$= \frac{WR^3}{EI}\int_0^{\pi/2} (1-\cos\theta)(\sin\theta)\,d\theta$$

$$= \frac{WR^3}{EI}\int_0^{\pi/2} (\sin\theta - \sin\theta\cos\theta)\,d\theta$$

$$= \frac{WR^3}{EI}\int_0^{\pi/2} (\sin\theta - \tfrac{1}{2}\sin 2\theta)\,d\theta$$

$$= \frac{WR^3}{EI}\left[-\cos\theta + \tfrac{1}{4}\cos 2\theta\right]_0^{\pi/2}$$

$$= \frac{WR^3}{2EI}$$

- $$\int \frac{y^2 ds}{EI} = \frac{2}{EI}\int_0^{\pi/2} (R\sin\theta)^2 \cdot (R \cdot d\theta)$$

$$= \frac{2R^3}{EI}\int_0^{\pi/2} \sin^2\theta \, d\theta$$

$$= \frac{2R^3}{EI}\int_0^{\pi/2} \tfrac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= \frac{2R^3}{EI}\left[\frac{\theta}{2} - \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{\pi R^3}{2EI}$$

- H:

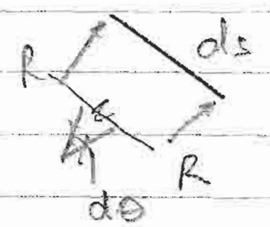
$$H = \int \frac{M_0 y \, ds}{EI} \Big/ \int \frac{y^2 ds}{EI}$$

$$= \frac{WR^3}{2EI} \Big/ \frac{\pi R^3}{2EI}$$

$$\Rightarrow \underline{H = \frac{W}{\pi}}$$

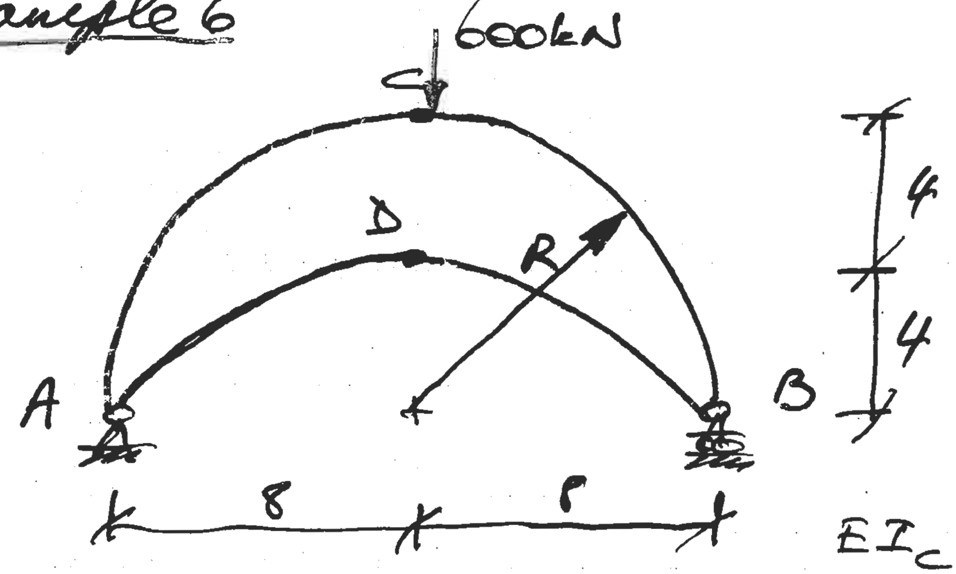
- NOTE THAT $ds = R\,d\theta$

ie



(ds LENGTH OF ARCH)

Example 6



$EI_c = 2 \times 10^6 \ kNm^2$

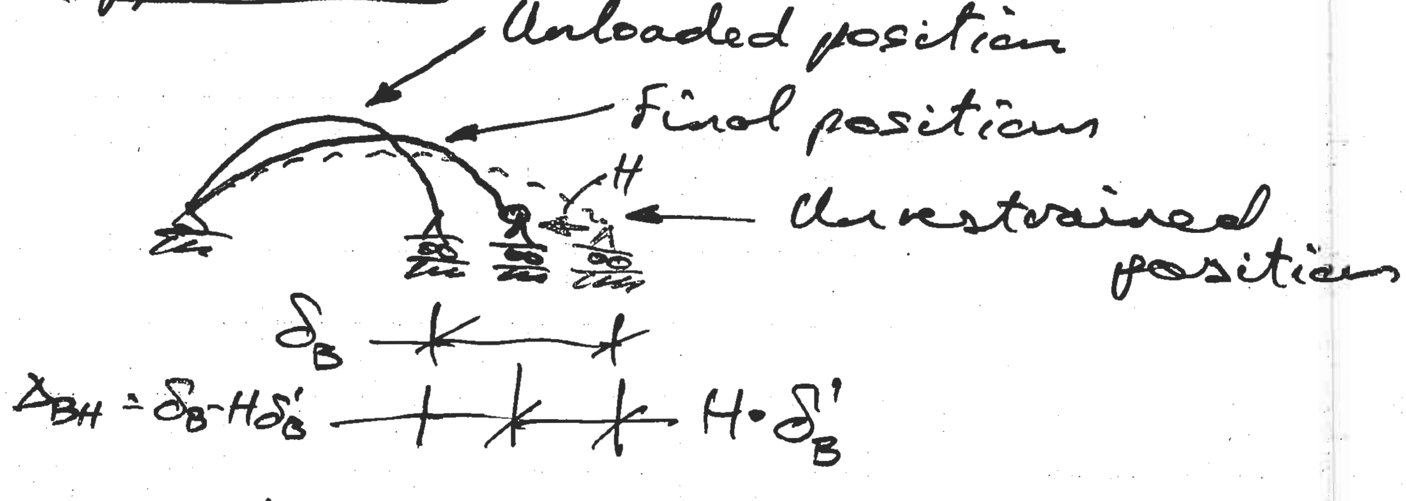
- ACB - Semi-circular, $I = I_c$, $R = 8$, $a = 0$
- ADB - Parabolic, $I = I_c \sec\theta$

Find,

a) Δ_{BH} b) BM at C and D.

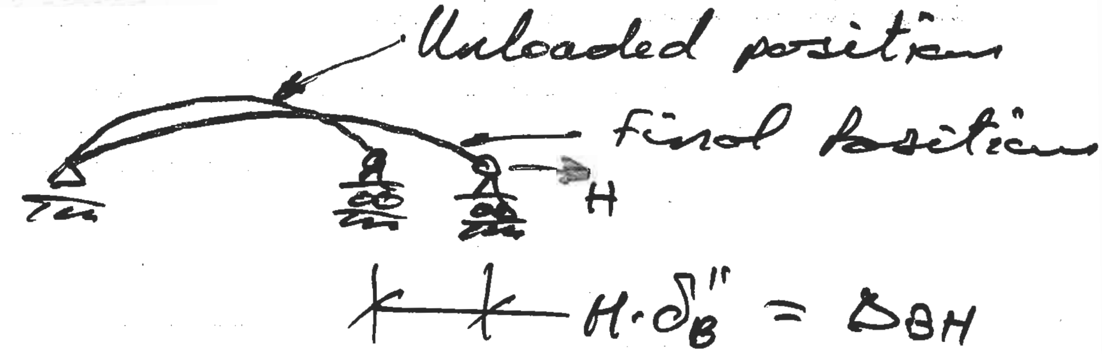
In this arch, the compatibility of displacement requirements are the same as Ex. 4.

Upper Arch:



δ_B & δ_B' can be calculated using Virtual Work

Lower Arch



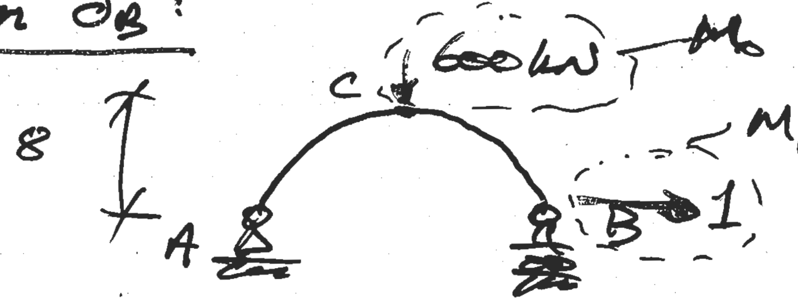
$$H \cdot \delta_B'' = \Delta_{BH}$$

Thus,

$$\Delta_{BH} = \delta_B - H\delta_B' = H\delta_B''$$

$$\Rightarrow \boxed{H = \frac{\delta_B}{\delta_B' + \delta_B''}}$$

Term δ_B:



$R = 8$
$\alpha = 0$

Ext. V.W. = Int. V.W.

$$\Rightarrow 1 \times \delta_B = \int \frac{M_0 y \, ds}{EI}$$

We can evaluate this similarly to Ex 5. Or, we note that if B were pinned we would have a $H = \frac{W}{\pi} = 600/\pi$ by Ex. 5. Note from Ex. 5 also that a unit load at B causes deflection of: $\Delta_{B1} = \frac{\pi R^3}{2EI}$

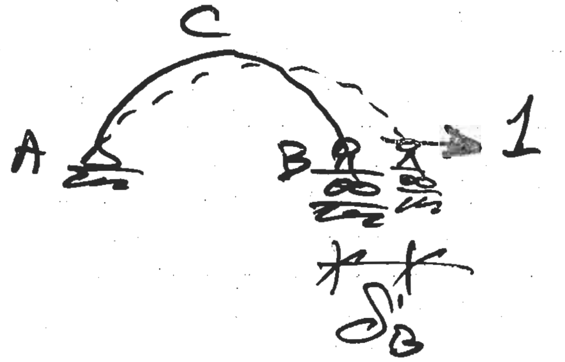
In this case the "push force" is not

a unit load but the released value of what would have been the horizontal reaction. Thus:

$$\delta_B = \frac{600}{\pi} \times \frac{\pi R^3}{2EI}$$

$$\Rightarrow \underline{\delta_B EI_c = 153.6 \times 10^3}$$

<u>Term δ_B'</u>



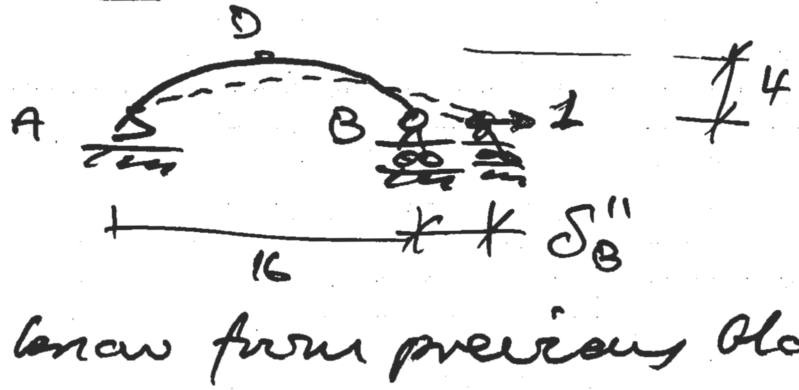
from Ex. 5, and above,

$$\delta_B' = \frac{\pi R^3}{2EI}$$

$$= \delta_B / (600/\pi)$$

$$\Rightarrow \underline{EI_c \delta_B \simeq 804.25}$$

<u>Term δ_B''</u>



We know from previous that

$$\delta_B'' = \int \frac{y^2 dx}{EI_c} = \frac{8h^2 L}{15 EI_c}$$

$$\Rightarrow \underline{EI_c \delta_B'' = 136.53}$$

Thus,

$$H = \frac{\delta_B}{\delta_B' + \delta_B''}$$

$$= \frac{153,600}{136.5 + 804.3}$$

$$\therefore \boxed{H = 163.26 \text{ kN}}$$

This is the interaction force between the arches & not a horizontal external reaction

Also,

$$\Delta_{BH} = H \cdot \delta_B'' = 163.26 \times \frac{136.53}{EI_c}$$

$$= 11.15 \text{ mm}$$

Further,

$$M_C = +1094 \text{ kNm}$$
$$M_D = -653 \text{ kNm}$$

} Tension bottom + Check ans.

ARCHES – THINGS TO NOTE

$$H = \frac{\int M_0 y \, ds/EI}{\int y^2 \, ds/EI}$$

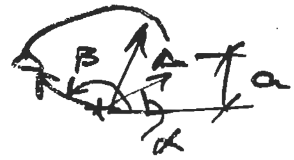
- $c = 4h/L^2$ $\quad [\, y = cx(L-x) \,]$
- H due to UDL $= \dfrac{wL^2}{8h}$ (h = height of parabolic Arch)

  Also, there is no moment in these types of arches.
- $\int y^2 ds/EI = \dfrac{8h^2L}{15EI}$ – for a parabolic Arch.
- Keep terms in H expression to ~~the~~ same power – really aids solution.
- Circular Arches:

  Any point P: $\quad x = \frac{L}{2} - R\cos\theta$

  $\quad\quad\quad\quad\quad y = R\sin\theta - a$

  where ~~as~~ a is:



  Integration limits are $\int_\alpha^\beta d\theta$.

COMPATIBILITY OF DISPLACEMENT IN ARCHES

Common feature – two "structures" with associated stiffness interact.

– there is one force transmitted between the structures. (H)

Solution:

For both structures, develop an expression for the final displacement in terms of H. [Equate expressions & solve for H.]

Expressions for final displacement generally consist of:

– disp. with no ~~restraint~~ External load – δ

– disp. for a unit load – δ'

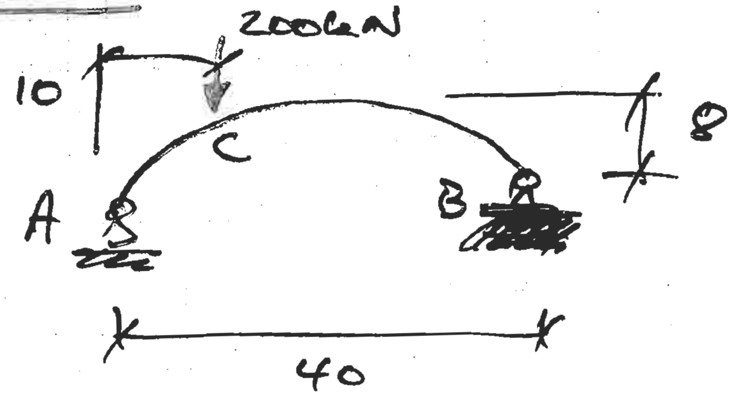
Thus for each structure we generally have: $\boxed{\Delta = \delta \pm H\delta'}$

If there is no load on a structure, only the force H, the expression will not contain δ.

Know enough to "customize" the above to the particular problem under consideration

Problems

(1)



$EI_c = 2\times10^6\ kNm^2$

$I = I_c \sec\theta$

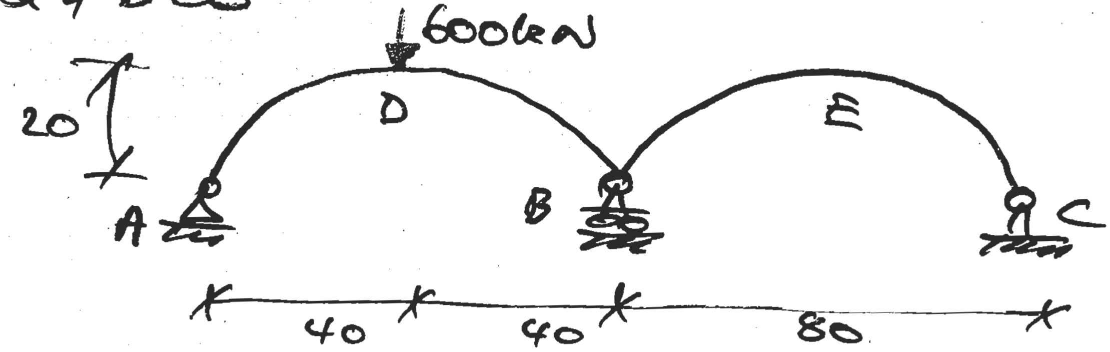
1) find H & M at crown — $H = 339.16kN$

2) if roller at B, find Δ_{BH} ? — 95mm

3) if B slips by 40mm, what is H ? $H = 80.56kN$

---

(2) Q4 S'00



$EI_c = 2\times10^7\ kNm^2$, $I = I_c \sec\theta$

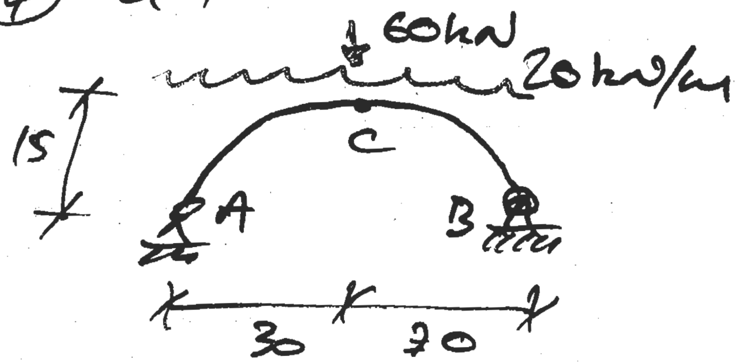
find Δ_{BH} and M_D & M_E

---

(3) Q4 S'89 → done in class

---

(4) Q4 S'98



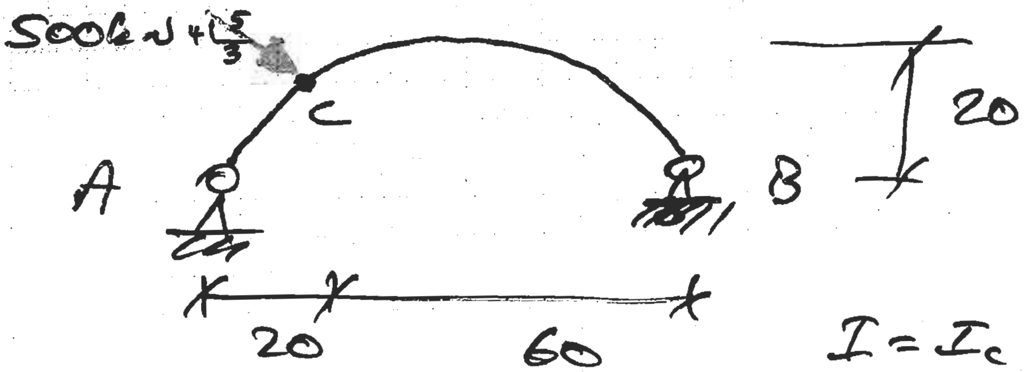
$E = 205\ kN/mm^2$

$I_c = 120\times10^7\ mm^4$

If $\Delta_{BH} = 40mm$ find H.

find M_{max} in the arch.

(5) Q 4 A '01



$I = I_c \sec\theta$

Find H & M_c